Intermodulation Products in Dual Carrier Transmission: Power Series Analysis

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The proposed Viking dual-carrier telemetry modes have generated an interest in determining the relative power levels of dual carrier intermodulation products (IMPs) for a Klystron amplifier. To this end, a finite-order power series model for the time domain input-output response of the Klystron was investigated. The parameters needed to define the model for a particular Klystron were derived from experimental measurements of the nonlinear power transfer characteristic of the amplifier for single carrier transmission. The power series approach, as described in this report, does not appear to be a useful analytic tool for predicting dual-carrier IMP levels. With the exception of the first-order IMP, the model is evidently too sensitive to small changes in the experimental single carrier data to provide accurate IMP information.

I. Introduction

In support of proposed Viking dual-carrier operations, a joint effort is being made to experimentally and analytically determine the relative power levels of dual carrier intermodulation products (IMPs) in a Klystron amplifier. This report addresses the analytical side of this investigation.

A common analytic approach to the problem of computing IMP power levels in nonlinear amplifiers, such as traveling wave tubes and Klystrons, is to use a power series model (Ref. 1). That is, the output $y(t)$ of the amplifier is assumed to be accurately approximated by a weighted sum of lower order powers of the input signal $x(t)$, within the amplifier passband. It can be shown that even powers of $x(t)$ do not produce components within the amplifier passband, and are therefore deleted in the power series model (Ref. 1). Accordingly, an Nth order power series approximation has the form

$$y(t) = \sum_{n=1}^{N} C_n x^{2n-1}(t), \tag{1}$$

with components of $y(t)$ outside the amplifier passband being ignored.

It can be demonstrated that the Nth order power series model above could, in principle, be used to generate the first $N - 1$ IMPs in dual carrier transmissions. The power level of the mth IMP was shown to consist of a weighted sum of the coefficients $C_{m+1}, C_{m+2}, \cdots, C_N$. It was argued that the C_n's could be derived from experimental measurements of the nonlinear power transfer characteristic of the amplifier for single carrier transmission. The calculation of the higher order coefficients C_n is increasingly sensitive to very small variations of this experimental power transfer characteristic. Noise limitations on the accuracy of the measured data dictate that N should not be too large (e.g., $N \leqq 5$); therefore, this analytical model only yields information about the lower order IMPs.

In this report, the power series model is applied to measured power transfer data for Viking channel 17 supplied by R. Leu. All necessary formulas are derived in the Appendix. In the next section, the step-by-step mathematical procedure used to calculate lower order IMP power levels based on the given experimental data is outlined in detail. The results of this analysis are presented graphically and discussed in the concluding section.

II. Procedure

We are given a set of M measured values of output power, P, in kilowatts versus input power, P_i, in milliwatts for a Klystron amplifier with a single carrier input. As indicated in Eq. (A-12) of the Appendix, it is convenient to use an Nth order power series model of the form

$$y(t) = \sum_{n=1}^{N} \frac{2^{2(n-1)}}{\binom{2n-1}{n-1}} E_n x^{2n-1}(t), \tag{2}$$

where $N \leqq M$, to relate the input $x(t)$ and output $y(t)$ of the amplifier, within its passband. Subject to this model, a single carrier input

$$x(t) = K \cos \omega t \tag{3}$$

at frequency ω, within the passband, produces a single carrier output

$$y(t) = L \cos \omega t \tag{4}$$

For convenience, assume K is expressed in volts, and L is in kilovolts; this choice of units will be justified later. Then, if the amplifier input resistance, R_i, and output resistance R_o are given in ohms, we have

$$\begin{aligned} P_i &= 500\, K^2/R_i \\ P_o &= 500\, L^2/R_o \end{aligned} \tag{5}$$

As shown in the Appendix, the relation between K and L is most conveniently given indirectly by

$$V = \sum_{n=1}^{N} E_n U^{n-1} \tag{6}$$

using the one-to-one mapping

$$\begin{aligned} U &\equiv K^2 \\ V &\equiv L/K \end{aligned} \tag{7}$$

with U in (volts)^2 and V in kilovolts/volt. This implies that E_n has units of kilovolts/(volt)$^{2n-1}$. Combining Eqs. (5) and (7), we can write

$$\begin{aligned} U &= P_i R_i/500 \\ V &= \sqrt{P_o R_o / P_i R_i} \end{aligned} \tag{8}$$

The particular Klystron amplifier analyzed in this report has input and output resistances of 50 and 377 Ω, respectively. Therefore, the M measured values of P_o versus P_i can be transformed into a set of M points

$$\{(U_k, V_k); \quad k = 1, 2, \cdots, M\}$$

We can now use a least-square approach to solve for the N coefficients E_n. Define matrices E, G, and H:

$$E \equiv \begin{pmatrix} E_1 \\ E_2 \\ \cdot \\ \cdot \\ E_N \end{pmatrix} \qquad H \equiv \begin{pmatrix} H_1 \\ H_2 \\ \cdot \\ \cdot \\ H_N \end{pmatrix}$$
$$G = \begin{pmatrix} G_{1,1} & G_{1,2} & \cdots & G_{1,N} \\ G_{2,1} & G_{2,2} & \cdots & G_{1,N} \\ \vdots & \vdots & & \vdots \\ G_{N,1} & G_{N,2} & \cdots & G_{N,N} \end{pmatrix} \tag{9}$$

where

$$\begin{aligned} G_{l,n} &\equiv \sum_{k=1}^{M} U_k^{l+n-2} \\ H_l &\equiv \sum_{k=1}^{M} U_k^{l-1} V_k \end{aligned} \tag{10}$$

It is proved in the Appendix that

$$E = G^{-1} H \tag{11}$$

so that we can determine the E_n's with the aid of a matrix inversion subroutine on the computer.

Let us briefly examine the units selected above. We are going to apply this technique to the following experimental data for Viking channel 17, supplied by R. Leu:

P_i, mW	P_o, kW
5.0	6.3
10.0	11.7
15.0	17.4
20.0	23.0
25.0	28.7
30.0	32.9
35.0	37.5
40.0	42.0
45.0	46.3
50.0	48.8

Thus, $M = 10$, and since R_i and R_o are 50 and 377 Ω, respectively,

$$\frac{1}{2} \leq U_k \leq 5$$

$$V_k \approx 3$$

The ranges of U_k and V_k above are sufficient to ensure that the elements of G and H lie within a range that allows the computer to perform the matrix inversion of Eq. (11) accurately, using double-precision arithmetic.

Having calculated the coefficients E_n according to the procedure above, we now want to compute the IMPs produced by the symmetric dual-carrier input

$$x(t) = A(\cos \omega_1 t + \cos \omega_2 t); \qquad \omega_2 > \omega_1 \tag{12}$$

where frequencies ω_1 and ω_2 lie near the single carrier frequency ω for channel 17. For the Nth order power series model, it is shown in the Appendix that the corresponding output has the form

$$y(t) = \sum_{m=0}^{N-1} B_m (\cos \alpha_m t + \cos \beta_m t)$$

where

$$B_m \equiv \sum_{n=m+1}^{N} E_n A^{2n-1} \binom{2n-1}{n-m-1}$$

$$\alpha_m \equiv \omega_1 - m(\omega_2 - \omega_1) \tag{13}$$

$$\beta_m \equiv \omega_2 + m(\omega_2 - \omega_1)$$

The output components at frequencies α_m and β_m have equal amplitudes B_m, since the input is composed of equal amplitude carriers. The output carrier components at frequencies ω_1 and ω_2 are identified by the index $m = 0$. Indices $m > 0$ denote the mth order IMP. Since $m = 10$, and the order N of our power series model must be less than or equal to M, we can, in principle, use the formulas above to investigate, at most, the first nine IMPs.

Now if A is in volts above, the units of E_n dictate that B_m is in kilovolts. The output carrier components each have power

$$P_c = 500\, B_0^2 / R_0 \tag{14}$$

expressed in kilowatts. The mth order IMPs are at power levels

$$P_m = 20 \log_{10} \left| \frac{B_m}{B_0} \right| \text{ dB} \tag{15}$$

relative to P_c.

III. Results

Before examining the dual-carrier IMP plots, based on experimental data for Viking channel 17, consider the accuracy of the analytic procedure that generated them. We have 10 measured values of output power P_o versus input power P_i for the single-carrier input case. The Nth order power series model of Eq. (2) defines a set of coefficients $\{E_n;\ n = 1, 2, \cdots, N\}$ that must be computed based on these data. This model defines a relation between P_i and P_o in terms of the E_n's that is most conveniently expressed by Eq. (6) in the domain (U, V); this new domain is a one-to-one mapping of the original domain (P_i, P_o), according to the specification of Eq. (8). The ten data points (P_i, P_o) are therefore converted into ten data points (U, V); then the E_n's are computed such that Eq. (6) is the least-square fit to the given data in the domain (U, V).

The simplest power series model of the form of Eq. (2) that produces dual-carrier IMPs is of order $N = 2$. It requires the computation of coefficients E_1 and E_2 from the given data, and yields information about the lowest order IMP. As shown in Fig. 1, the curve specified by Eq. (6) in terms of the computed values of E_1 and E_2 is a straight line in the domain (U, V). This curve has a square error $R_2 = 5.39 \times 10^{-3}$ with respect to the ten data points; visually, it does not appear to be a particularly tight fit to the given data. However, mapped into the domain (P_i, P_o), as presented in Fig. 2, the second-order power series model seems to correlate rather well with the experimental data. In fact, it is conceivable that many of the deviations between the data points (P_i, P_o) and the derived curve are of the order of the RMS errors in the measurement of the data. If this is indeed the case, then any further refinements in the derived curve by resorting to higher order power series models will be strongly influenced by the noise inherent in the experimental data. Since an Nth order model produces the first $N - 1$ dual carrier IMPs, this implies that information about the mth IMP determined according to this procedure decreases in accuracy quite rapidly as m increases. With this understanding, results are only presented below for models of order $N = 2, 3$, and 4, although models of higher order (up to $N = 10$) were also investigated.

To demonstrate the degree of refinement afforded by higher order models, results for $N = 4$ are given in the (U, V) and (P_i, P_o) domains in Figs. 3 and 4. For example, a comparison of Graphs 2 and 4 shows that the derived curves for $N = 2$ and $N = 4$ differ noticeably only in the high power region ($P_o \gtrsim 35$ kW). In the low-power region, the models of order $N = 2, 3$, and 4 should produce similar curves of output carrier power P_c, and first IMP power P_1, versus input dual-carrier amplitude A; this is evident in Figs. 5–7.

Figures 5–7 are replotted in Figs. 8–10, eliminating the parameter A. The experimental P_o versus P_i characteristic in Figs. 2 and 4 becomes more nonlinear with increasing input power. Consequently, as the input power is increased, a higher proportion of the output power is transferred from the carriers to the IMPs, as shown in Figs. 5 to 10. For $P_c < 13$ kW, the 2nd, 3rd, and 4th order computations of relative IMP power P_1 are consistent; for example, when the output carrier powers are each 10 kW, Figs. 8–10 indicate that the dual first-order IMPs are at a power level 34 dB below P_c. It was argued above that results for the 2nd and 3rd order IMPs are highly suspect. Indeed, Figs. 9 and 10 yield very different 2nd order IMP powers P_2, and the dip in P_2 at $P_c \approx 12$ kW is totally unexpected.

In summation, the analytic evaluation of dual-carrier IMP powers using a power series model is not a productive area of investigation. With the exception of the first-order IMP, accurate results cannot be obtained using the experimental data supplied.

Appendix

Calculation of IMPs for Symmetric Dual-Carrier Transmission Using Power Series Model

Consider the following power series model for a nonlinear amplifier. The output signal $y(t)$ is assumed to consist of a weighted sum of lower order powers of the input $x(t)$:

$$y(t) = \sum_{n=1,2,\cdots} [C_n x^{2n-1}(t) + D_n x^{2n}(t)] \qquad \text{(A-1)}$$

For convenience, the odd and even powers of $x(t)$ are separated in Eq. (A-1). It is assumed that the frequency response of the amplifier is flat over a narrow passband: components of the summation in Eq. (A-1) that fall outside this passband are neglected.

In the mathematical analysis below, the following two identities are needed:

$$\cos^{2n-1}\omega t = 2^{-2(n-1)} \sum_{m=0}^{n-1} \binom{2n-1}{n-m-1} \cos(1+2m)\,\omega t \qquad \text{(A-2)}$$

$$\cos^{2n}\omega t = 2^{-2n}\left[\binom{2n}{n} + 2\sum_{m=0}^{n-1}\binom{2n}{n-m-1}\cos(1+m)\,2\omega t\right] \qquad \text{(A-3)}$$

As suggested by S. Butman, these identities are easily proved. For example, consider Eq. (A-2):

$$\cos^{2n-1}\omega t = \left[\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right]^{2n-1}$$

$$= 2^{-(2n-1)} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} e^{j(2n-2\ell-1)\omega t}$$

(binomial expansion)

$$= 2^{-(2n-1)} \sum_{\ell=0}^{n-1} \binom{2n-1}{\ell} 2\cos(2n-2\ell-1)\,\omega t$$

(combining terms pairwise)

$$= 2^{-2(n-1)} \sum_{m=0}^{n-1} \binom{2n-1}{n-m-1} \cos(1+2m)\,\omega t$$

$$(m = n - \ell - 1)$$

And Eq. (A-3) is similarly proved.

Consider a single-carrier input signal

$$x(t) = K\cos\omega t \qquad \text{(A-4)}$$

where the frequency ω lies within the amplifier passband. It is evident from Eq. (A-3) that $x^{2n}(t)$ produces components at even harmonics of ω, which fall beyond the amplifier passband by assumption. From Eq. (A-2), it is seen that $x^{2n-1}(t)$ yields components at odd harmonics of ω; it is assumed that only those components at the fundamental frequency ω are passed by the amplifier. Therefore, the output is given by

$$y(t) = L\cos\omega t \qquad \text{(A-5)}$$

where the output amplitude L is a weighted sum of the odd powers of the input amplitude K:

$$L = \sum_{n=1,2,\cdots} E_n K^{2n-1} \qquad \text{(A-6)}$$

and

$$E_n \equiv 2^{-2(n-1)} \binom{2n-1}{n-1} C_n \qquad \text{(A-7)}$$

Next, suppose the input signal is the sum of two equal amplitude (symmetric) carriers at frequencies ω_1 and ω_2, both within the amplifier passband:

$$x(t) = A(\cos\omega_1 t + \cos\omega_2 t) = 2A\cos\omega_d t\cos\omega_0 t \qquad \text{(A-8)}$$

where

$$\omega_d \equiv \frac{1}{2}(\omega_2 - \omega_1)$$

$$\omega_0 \equiv \frac{1}{2}(\omega_1 + \omega_2)$$

and, without loss of generality, $\omega_2 > \omega_1$. Since ω_1 and ω_2 lie within the passband, so must ω_0. Also, assume ω_1 is near ω_2 such that $\omega_d << \omega_0$. Even terms $x^{2n}(t)$ contain the factor $\cos^{2n}\omega_0 t$, which falls outside the passband. Therefore we only need to consider the odd terms $x^{2n-1}(t)$ in Eq. (A-1):

$$\cos^{2n-1}\omega_0 t = 2^{-2(n-1)}\binom{2n-1}{n-1}\cos\omega_0 t$$

$$+ \text{ higher order harmonics of } \omega_0$$

$$\cos^{2n-1}\omega_d t = 2^{-2(n-1)}\sum_{m=0}^{n-1}\binom{2n-1}{n-m-1}\cos(1+2m)\,\omega_d t$$

Therefore

$$x^{2n-1}(t) = A^{2n-1}\,2^{-2(n-1)}\binom{2n-1}{n-1}$$

$$\times \sum_{m=0}^{n-1}\binom{2n-1}{n-m-1}\underbrace{2\cos(1+2m)\,\omega_d t\cos\omega_0 t}_{\cos\alpha_m t + \cos\beta_m t}$$

where

$$\begin{aligned} \alpha_m &\equiv \omega_1 - m(\omega_2-\omega_1) \\ \beta_m &\equiv \omega_2 + m(\omega_2-\omega_1) \end{aligned} \qquad \text{(A-9)}$$

Substituting into Eq. (A-1),

$$y(t) = \sum_{n=1,2,\cdots} C_n A \exp 2n - 1_2 - 2(n-1)\binom{2n-1}{n-1}$$

$$\times \sum_{m=0}^{n-1}\binom{2n-1}{n-m-1}(\cos\alpha_m t + \cos\beta_m t)$$

Interchanging the order of summation above,

$$y(t) = \sum_{m=0,1,\cdots} B_m(\cos\alpha_m t + \cos\beta_m t) \qquad \text{(A-10)}$$

where

$$B_m \equiv \sum_{n=m+1,m+2,\cdots} E_n A^{2n-1}\binom{2n-1}{n-m-1} \qquad \text{(A-11)}$$

and E_n has been defined earlier. Because the input consists of symmetric dual carriers, the output components at frequencies α_m and β_m have the same amplitude B_m, as shown in Eq. (A-10). The index $m = 0$ above denotes the output carrier components; indices $m > 0$ denote the mth order IMPs at frequencies α_m and β_m.

Suppose we are given a set of M experimental values of output power P_o versus input power P_i for the single carrier case defined by Eqs. (A-4) to (A-7). In order to use these measured data to extract information about the lower order IMPs, we must first compute a set of E_n's. Since we are only given M independent measurements to work with, we can determine at most M unique coefficients E_n. Combining Eqs. (A-1) and (A-7), we therefore specialize to an Nth order power series model

$$y(t) = \sum_{n=1}^{N}\frac{2^{2(n-1)}}{\binom{2n-1}{n-1}}E_n x^{2n-1}(t) \qquad \text{(A-12)}$$

where $N \leqq M$. The even power terms $x^{2n}(t)$ in Eq. (A-1) have been deleted above, since they are irrelevant to the problem of interest.

Specializing Eq. (A-6) to the Nth order model, the single carrier input amplitude K is related to the output amplitude L by

$$L = \sum_{n=1}^{N} E_n K^{2n-1} \qquad \text{(A-13)}$$

For convenience, Eq. (A-13) can be transformed using the one-to-one mapping

$$\begin{aligned} U &\equiv K^2 \\ V &\equiv L/K \end{aligned} \qquad \text{(A-14)}$$

leading to

$$V = \sum_{n=1}^{N} E_n U^{n-1} \qquad \text{(A-15)}$$

Since

$$P_i \,\alpha\, K^2$$

$$P_o \,\alpha\, L^2$$

it follows that

$$U \,\alpha\, P_i$$

$$V \,\alpha\, \sqrt{P_o/P_i}$$

Therefore, the M experimental values of P_o versus P_i can be mapped into a set of M points

$$\{(U_k, V_k);\quad k = 1, 2, \cdots, M\}$$

For $N = M$, we can determine a unique set of M coefficients E_n by solving M simultaneous linear equations

$$\sum_{n=1}^{M} U_k^{n-1} E_n = V_k; \qquad k = 1, 2, \cdots, M \tag{A-16}$$

In general, for $N \leqq M$, a least-square approach can be used: that is, we want to evaluate the unique set of N values $\{E_n; \quad n = 1, 2, \cdots, N\}$ such that the $(N-1)$th order polynomial

$$F_N(U) \equiv \sum_{n=1}^{N} U^{n-1} E_n \tag{A-17}$$

is the (unique) least square fit to the M experimental points $\{(U_k, V_k); \quad k = 1, 2, \cdots, M\}$. Thus, we want to choose the E_n's to minimize the square error

$$R_N \equiv \sum_{k=1}^{M} [V_k - F_N(U_k)]^2 \tag{A-18}$$

This minimization implies a set of N constraints

$$\frac{\partial R_N}{\partial E_\ell} = \sum_{k=1}^{M} 2\left[V_k - \sum_{n=1}^{N} U_k^{n-1} E_n\right](-U_k^{\ell-1}) = 0;$$

$$\ell = 1, 2, \cdots, N \tag{A-19}$$

Eq. (A-19) can be simplified to yield N simultaneous linear equations that can be solved for the N desired coefficients E_n:

$$\sum_{n=1}^{N} G_{\ell,n} E_n = H_\ell; \qquad \ell = 1, 2, \cdots, N$$

where

$$\left.\begin{aligned} G_{\ell,n} &\equiv \sum_{k=1}^{M} U_k^{\ell+n-2} \\ H_\ell &\equiv \sum_{k=1}^{M} U_k^{\ell-1} V_k \end{aligned}\right\} \tag{A-20}$$

The N-fold linear system of Eq. (A-20) can be expressed in matrix form:

$$\underset{N\times N}{[G_{\ell,n}]}\ \underset{N\times 1}{[E_n]} = \underset{N\times 1}{[H_\ell]}$$

Then, the E_n's can be deduced with the aid of a computer using a matrix inversion subroutine:

$$[E_n] = [G_{\ell,n}]^{-1} [H_\ell] \tag{A-21}$$

Note that for $N = M$, the E_n's defined by Eq. (A-16) yield a square error R_M of zero. Therefore these E_n's must be identical to those computed using the least-square approach of Eq. (A-21) when $N = M$.

Reference

1. Putz, J. L., "Predicting Nonlinear Effects in TWT's," *Microwaves*, June 1965.

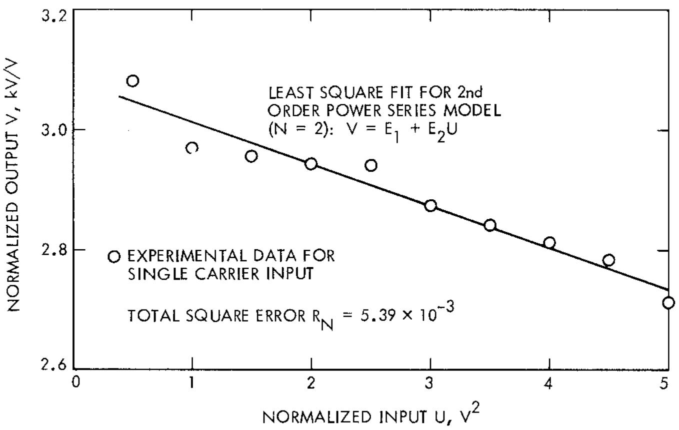

Fig. 1. Determination of 2nd order ($N = 2$) power series model

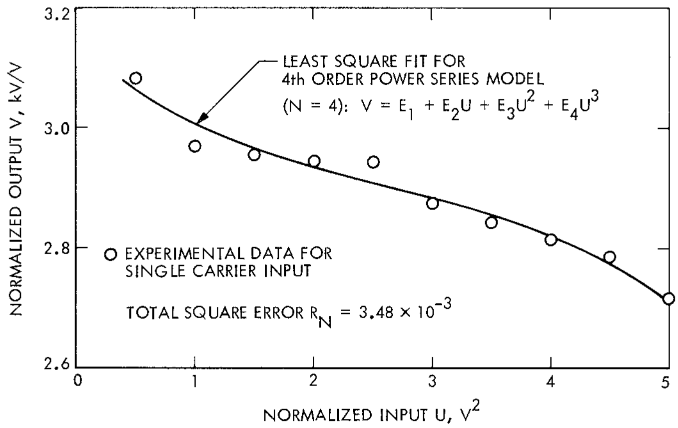

Fig. 3. Determination of 4th order ($N = 4$) power series model

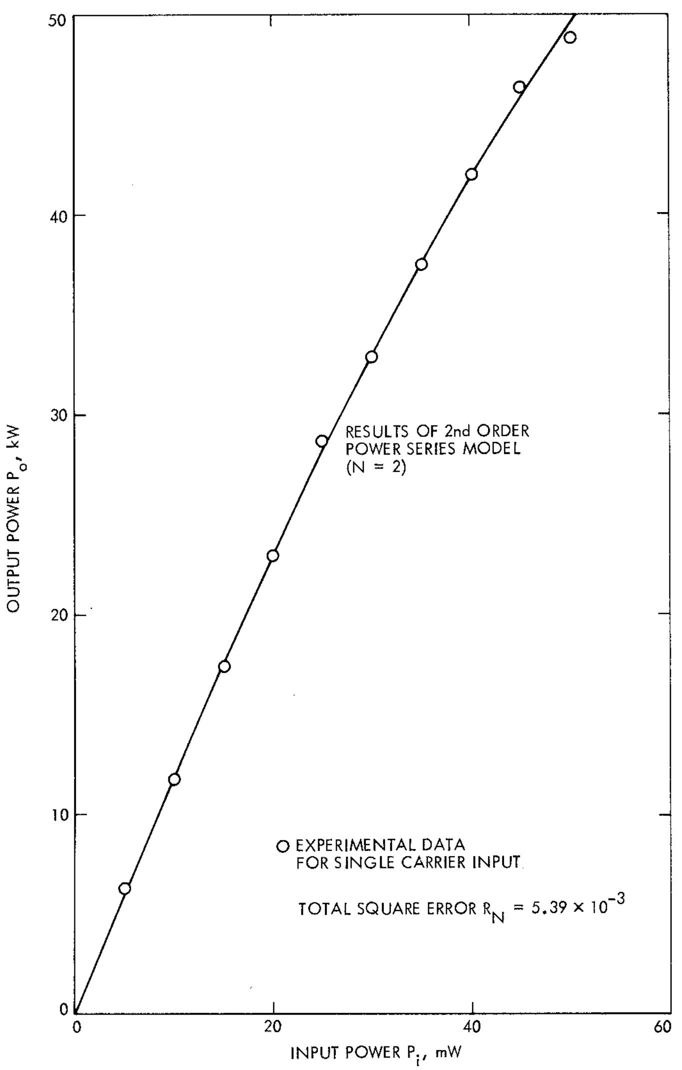

Fig. 2. Transformation of Fig. 1 into input/output power domain

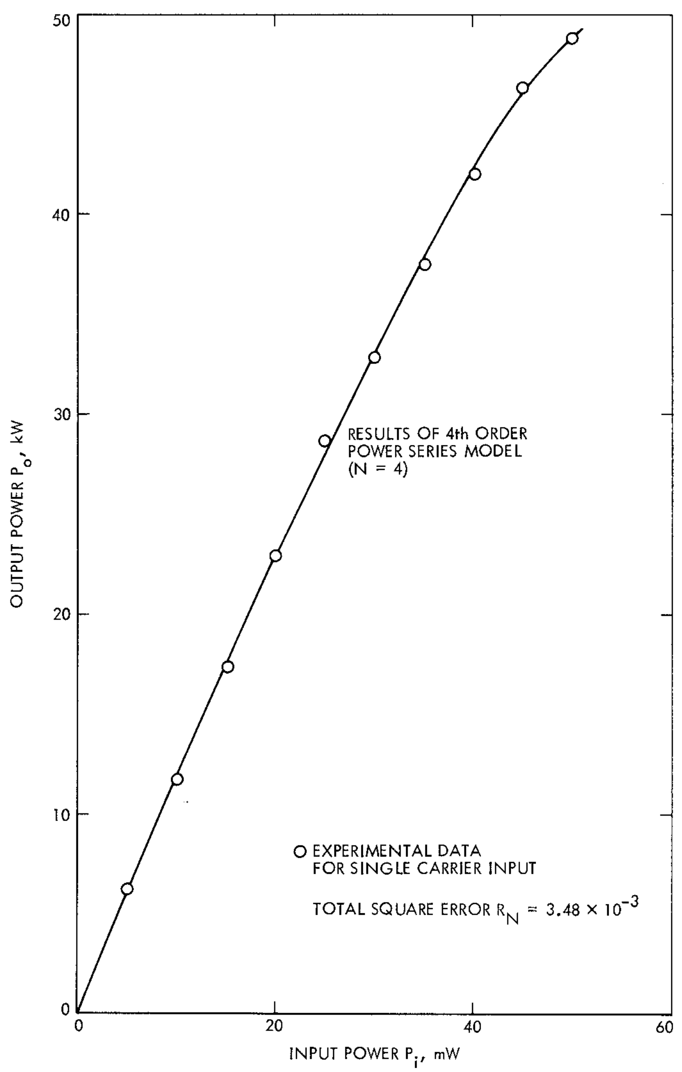

Fig. 4. Transformation of Fig. 3 into input/output power domain

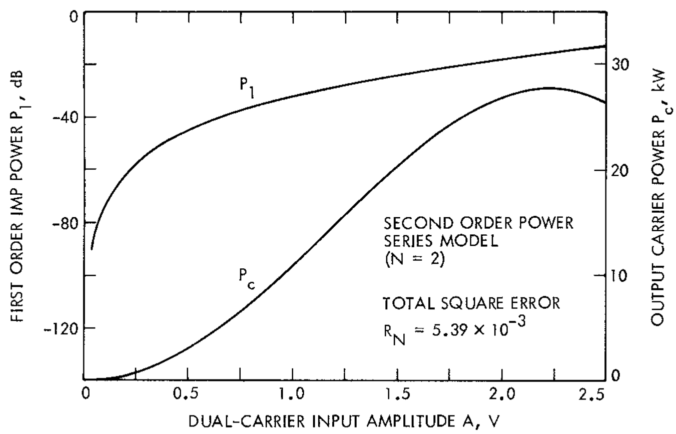

Fig. 5. Theoretical dual-carrier input results for 2nd order power series model

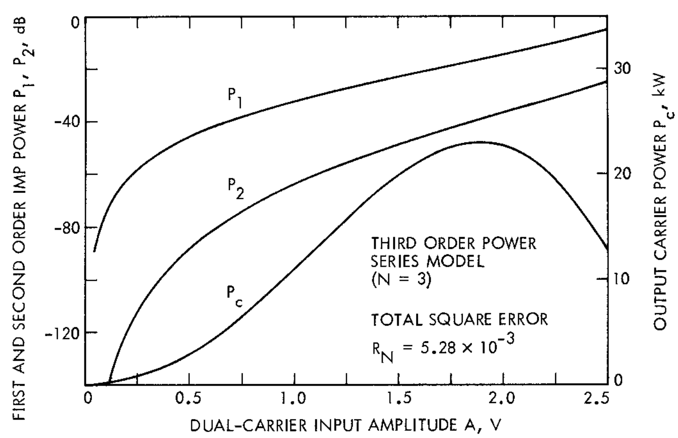

Fig. 6. Theoretical dual-carrier input results for 3rd order power series model

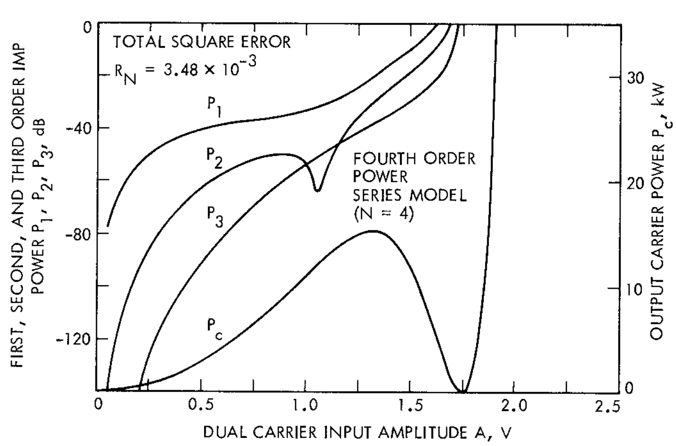

Fig. 7. Theoretical dual-carrier input results for 4th order power series model

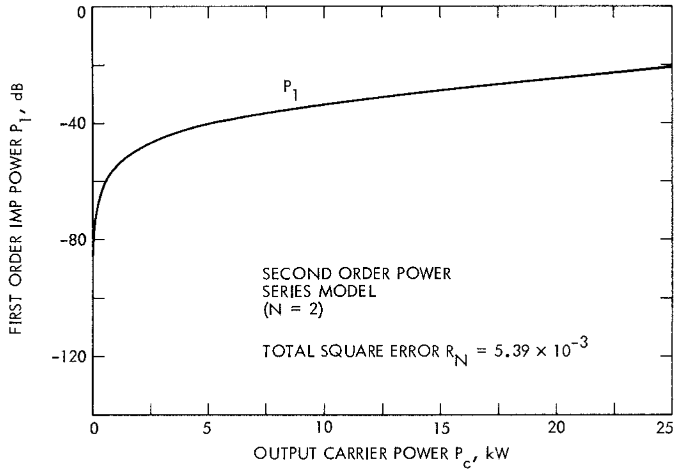

Fig. 8. Behavior of 1st order IMP power as a function of output carrier power

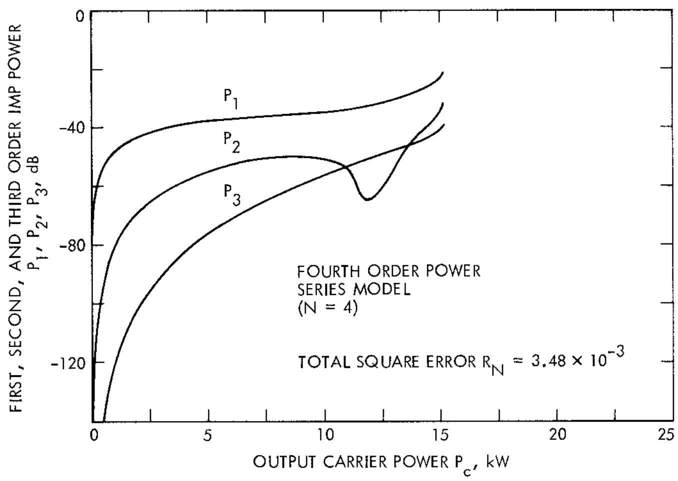

Fig. 10. Behavior of 1st, 2nd, and 3rd order IMP powers as a function of output carrier power

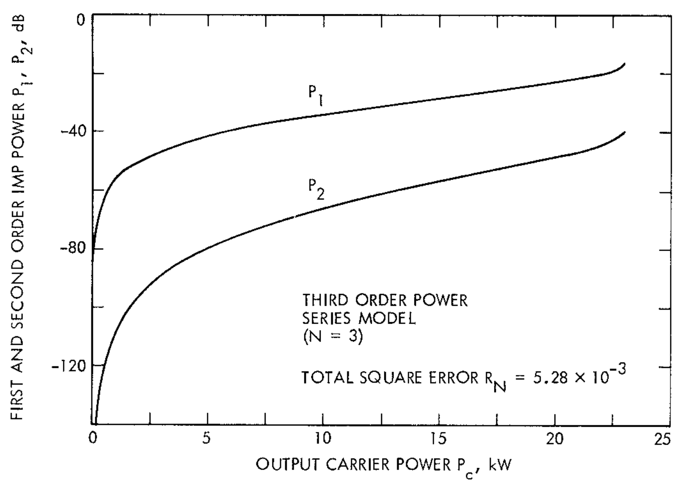

Fig. 9. Behavior of 1st and 2nd order IMP power as a function of output carrier power